

A constructive version of the Boyle-Handelman theorem on the spectra of nonnegative matrices

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Abstract

A constructive version of the celebrated Boyle-Handelman theorem on the non-zero spectra of nonnegative matrices is presented.

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1 Introduction.

Let

$$\sigma = (\lambda_1, \dots, \lambda_n)$$

be a list of complex numbers and let

$$s_k := \lambda_1^k + \dots + \lambda_n^k, \quad k = 1, 2, 3, \dots$$

The nonnegative inverse eigenvalue problem (**NIEP**) asks for necessary and sufficient conditions on σ in order that it be the spectrum of an entry-wise nonnegative matrix. If this occurs, we say that σ is *realizable*, and we call a nonnegative matrix A with spectrum σ a *realizing matrix* for σ .

A necessary condition for realizability coming from the Perron-Frobenius theorem [2] is that there exists j with λ_j real and $\lambda_j \geq |\lambda_i|$, for all i . Such a λ_j is called the *Perron root* of σ .

A more obvious necessary condition is that all the s_k are nonnegative. A stronger form of this condition was found independently by Loewy and London [11] and Johnson [8], namely:

$$(\mathbf{JLL}) \quad n^{k-1}s_{km} \geq s_m^k, \text{ for all positive integers, } k \text{ and } m.$$

In terms of n , a complete solution of the NIEP is available only for $n \leq 4$. The solution for $n = 4$, expressed in terms of inequalities for the s_k , appears in the PhD thesis of Meehan [12] and a solution in terms of the coefficients of the characteristic polynomial has been published more recently by Torre-Mayo, Abril-Raymundo, Alarcia-Estevez, Marijuan, and Pisanero [14].

However, the same problem in which we may augment the list σ by adding an arbitrary number N of zeros was solved by Boyle and Handelman [4]. Using a range of tools coming from linear algebra, dynamical systems, ergodic theory, and graph theory, they proved the remarkable result that if

1. σ has a Perron element $\lambda_1 > |\lambda_j|$ (all $j > 1$) and
2. $s_k \geq 0$ for all positive integers k (and $s_m = 0$ for some m implies $s_d = 0$ for all positive divisors d of m), then

$$\sigma_N := (\lambda_1, \dots, \lambda_n, 0, \dots, 0) \text{ (} N \text{ zeros)}$$

is realizable for all sufficiently large N .

Under these assumptions, a realizing matrix can be chosen to be primitive. See Friedland [6] for an extension to the irreducible case.

The proof of the Boyle-Handelman result is not constructive and does not provide a bound on the minimal number $N = N(\sigma)$ of zeros required for realizability.

Finding a constructive proof, with a bound on the minimum number N of zeros required, has been an area of much research, and a number of special cases have been resolved. In particular, a best possible result in the case that $\operatorname{Re}(\lambda_j) \leq 0$, for all $j > 1$, has been obtained by Šmigoc and the author [9] and, when σ is real and has exactly two positive entries, a constructive proof with a bound on N has also been found [10].

In the case that σ is real and has just one positive entry, then the inequality $s_1 \geq 0$ is necessary and sufficient for realizability. This was proved by Suleimanova [13] and this is often viewed as the first result on the NIEP. Friedland [5] re-proved her result by showing that the companion matrix with spectrum σ has nonnegative entries, and matrices related to companion matrices are used in the cited work with Šmigoc.

Here, a constructive approach to the Boyle-Handelman result is presented. It is shown that a certain kind of patterned matrix is "universal" for the realization of spectra with power sums $s_k > 0$, ($k = 1, 2, 3, \dots$). in the sense that all such spectra satisfying the Perron condition (i) above can, with sufficiently many zeros added, be realized as the spectrum of a primitive nonnegative matrix with that pattern.

2 A matrix related to Newton's identities

Let

$$\begin{aligned}\tau &= (\mu_1, \dots, \mu_n), \\ x_k &:= \mu_1^k + \dots + \mu_n^k, \quad k = 1, 2, 3, \dots \\ q(x) &:= \prod_{i=1}^n (x - \mu_i) \\ &= x^n + q_1 x^{n-1} + \dots + q_n.\end{aligned}$$

Let $X_n =$

$$\begin{pmatrix} x_1 & 1 & 0 & \dots & \dots & \dots & 0 \\ x_2 & x_1 & 2 & 0 & \dots & \dots & 0 \\ x_3 & x_2 & x_1 & 3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n-1} & x_{n-2} & \dots & \dots & \dots & x_2 & x_1 & n-1 \\ x_n & x_{n-1} & \dots & \dots & \dots & x_3 & x_2 & x_1 \end{pmatrix}$$

The matrix X_n occurs in the context of the Newton identities relating the coefficients of a polynomial to the power sums of its roots. If we use Cramer's rule to express the q_i in terms of the x_j , we get

$$\det(X_n) = (-1)^n n! q_n.$$

However, the matrix X_n itself, as distinct from its determinant, does not appear to have been widely investigated. A key observation is:

Proposition 1 *The characteristic polynomial of X_n is*

$$Q(x) = x^n + nq_1 x^{n-1} + n(n-1)q_2 x^{n-2} + \dots + n!q_n.$$

Since X_n has nonnegative entries if the x_i are nonnegative, it follows that the spectrum of $Q(x)$ is realizable if the x_i , ($i = 1, 2, \dots, n$), are nonnegative.

Suppose that we are given a list $\sigma = (\lambda_1, \dots, \lambda_n)$ that we wish to realize as the spectrum of a nonnegative matrix.

Let

$$\begin{aligned}f(x) &:= \prod_{i=1}^n (x - \lambda_i) \\ &= x^n + p_1 x^{n-1} + \dots + p_n.\end{aligned}$$

Let

$$q(x) := x^n + q_1 x^{n-1} + \dots + q_n$$

where

$$q_i = \frac{p_i}{n(n-1) \dots (n-i+1)}, \quad i = 1, 2, \dots, n.$$

Then the corresponding $Q(x)$ is $f(x)$. Now the power sums x_i of the roots of $q(x)$ are nonnegative if and only if that holds for

$$x^n + nq_1 x^{n-1} + \dots + n^n q_n.$$

Hence we have

Theorem 2 σ is realizable by the matrix X_n if the j th power sum of the roots of the polynomial

$$J_n(f(x)) := x^n + p_1 x^{n-1} + \frac{n}{n-1} p_2 x^{n-2} + \frac{n^2}{(n-1)(n-2)} p_3 x^{n-3} + \dots + \frac{n^{n-1}}{(n-1)!} p_n$$

is nonnegative for $j = 1, 2, 3, \dots, n$.

But now suppose that we choose $N > n$ and ask for the realizability of σ with $N - n$ zeros added. This amounts to replacing $f(x)$ by $x^{N-n} f(x)$ and $J_n(f(x))$ by $x^{N-n} J_N(f(x))$, where

$$J_N(f(x)) := x^N + p_1 x^{N-1} + \frac{N}{N-1} p_2 x^{N-2} \quad (1)$$

$$+ \frac{N^2}{(N-1)(N-2)} p_3 x^{N-3} + \dots + \frac{N^{N-1}}{(N-1)(N-2) \dots (N-n+1)} p_n. \quad (2)$$

So σ with $N - n$ zeros added is realizable by the matrix X_N if the j th power sums of the roots of the polynomial $J_N(f(x))$ are nonnegative for $j = 1, 2, 3, \dots, N$.

But observe that as $N \rightarrow \infty$, $J_N(f(x)) \rightarrow f(x)$, since n is fixed.

Suppose that the power sums s_j of the elements of σ are positive for all $j \geq 1$. Then, on continuity grounds, one might expect that for sufficiently large N , the power sums of the roots of $J_N(f(x))$ would also be positive. However, this is not true in general, but it is true if σ has its Perron element

$$\lambda_1 > |\lambda_j| \quad (j = 1, 2, \dots, n).$$

In this case, σ with sufficiently many zeros added is the spectrum of a non-negative matrix X_N .

Since, we only require that the j th power sum of the roots of $J_N(f(x))$ be nonnegative for $j = 1, 2, \dots, N$, one can obtain a bound on the minimal number of zeros required.

3 Main Theorem

We now state the main result of this paper.

Theorem 3 *Let*

$$\sigma = (\lambda_1, \dots, \lambda_n),$$

be a list of complex numbers with corresponding power sums

$$s_k := \lambda_1^k + \dots + \lambda_n^k, \quad k = 1, 2, 3, \dots$$

Suppose that

- (i) $|\lambda_1| > |\lambda_j|$, (all $j > 1$)
- (ii) $s_1 \geq 0$, and $s_m > 0$, for all $m \geq 2$.

Let

$$\begin{aligned} f(x) &= \prod_{i=1}^n (x - \lambda_i) \\ &= x^n + p_1 x^{n-1} + \dots + p_n. \end{aligned}$$

$$\gamma = 2 \max(1, |p_1|, |p_2|^{1/2}, \dots, |p_n|^{1/n}).$$

$$\begin{aligned} \lambda_0 &= \max\{|\lambda_j| : j > 1\}, \\ R &= \frac{(\lambda_1 - \lambda_0)}{4}, \quad \ell = \frac{3\lambda_1 + \lambda_0}{\lambda_1 + 3\lambda_0}, \quad r = \min(R, 1), \end{aligned}$$

$$m = \max\{1, \lambda_1\}, \quad N_0 = \left\lceil \frac{\ln(2n-2)}{\ln(\ell)} \right\rceil,$$

$$M = \min\{1, s_2, \dots, s_{N_0}\},$$

and

$$N = \left\lceil 2 \left(\frac{16\gamma n N_0 (m+r)^{N_0-1}}{3^{1/2} M r} \right)^n \right\rceil.$$

Then σ with $N - n$ zeros added is the spectrum of the nonnegative matrix X_N , with $x_k := \mu_1^k + \dots + \mu_n^k$, $k = 1, 2, 3, \dots, N$, where

$$J_N(f(x)) = (x - N\mu_1)(x - N\mu_2) \dots (x - N\mu_n).$$

Given a list σ satisfying the hypotheses, it is relatively easy to find N for which $J_N(f(x))$ has the corresponding power sums nonnegative, so one obtains a reasonably efficient constructive algorithm. However, the number of zeros required in the construction is not optimal in general.

4 Proofs of the results

Let $P =$

$$\begin{pmatrix} 1 & 0 & 0 & & \cdot & \cdot & \cdot & & 0 \\ q_1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & & 0 \\ q_2 & q_1 & \frac{1}{2} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ q_3 & q_2 & \frac{q_1}{2} & \frac{1}{6} & 0 & \cdot & & & \\ \cdot & \cdot & \frac{q_2}{2} & \frac{q_1}{6} & \frac{1}{24} & \cdot & & & \\ \cdot & & & \frac{q_2}{6} & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & & \\ q_{n-2} & & & & & & & \cdot & \\ q_{n-1} & q_{n-2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{(n-2)!} & 0 \\ q_n & q_{n-1} & \frac{q_{n-2}}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{q_1}{(n-2)!} & \frac{1}{(n-1)!} \end{pmatrix}$$

and let $C =$

$$\begin{pmatrix} 0 & 1 & 0 & & \cdot & \cdot & \cdot & & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & & 0 \\ \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & & \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & & \\ \cdot & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & & & & \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 1 \\ -n!q_n & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -nq_1 \end{pmatrix}$$

be the companion matrix of $Q(x) = x^n + nq_1x^{n-1} + n(n-1)q_2x^{n-2} + \dots + n!q_n$.

Direct multiplication, using the Newton identities, yields $PC = X_nP$. This proves the proposition.

To obtain the desired bound we use the following refinement by Bhatia, Elsner and Krause [3] of a classical result of Ostrowski.

Theorem 4 *Let $f(x) = x^n + a_1x^{n-1} + \dots + a_n$ and $g(x) = x^n + b_1x^{n-1} + \dots + b_n$ be real polynomials with roots $\alpha_1, \dots, \alpha_n$ and β_1, \dots, β_n , respectively. Then there is a labelling of β_1, \dots, β_n such that*

$$\max\{|\alpha_i - \beta_i| : 1 \leq i \leq n\} \leq \left(\frac{16}{3\sqrt{3}}\right) \left(\sum_{k=1}^n |a_k - b_k| \gamma^{n-k}\right)^{1/n},$$

where $\gamma = 2 \max\{|a_k|^{1/k}, |b_k|^{1/k} : 1 \leq k \leq n\}$.

[The original Ostrowski result had the factor $(2n-1)$ in place of $\left(\frac{16}{3\sqrt{3}}\right)$].

Now let

$$\begin{aligned} f(x) &= (x - \lambda_1) \dots (x - \lambda_n) \\ &= x^n + p_1 x^{n-1} + \dots + p_n \end{aligned}$$

and

$$g(x) = x^n + p_1 x^{n-1} + \left(\frac{N}{N-1} \right) p_2 x^{n-2} + \dots + \left(\frac{N^{n-1}}{(N-1)\dots(N-n+1)} \right) p_n.$$

We note that if $g(x)$ has nonnegative Newton power sums, then the corresponding matrix X_N is nonnegative and has spectrum $N\lambda_1, \dots, N\lambda_n$.

Suppose that $\lambda_1 > |\lambda_j|$ (all $j > 1$) and let $\lambda_0 = \max(|\lambda_j| : j = 2, \dots, n)$ and $R = \frac{\lambda_1 - \lambda_0}{4}$. Let $\ell = \frac{\lambda_0 + R}{\lambda_1 - R}$, so $\ell < 1$. Let $r = \min\{1, R\}$. Let

$$s_k = \lambda_1^k + \dots + \lambda_n^k$$

for $k = 1, 2, \dots$. Assume that $s_1 \geq 0$, and that $s_k > 0$, for all $k > 1$. Let

$$M = \min\{1, s_k : k = 2, 3, \dots\}.$$

Let μ_1, \dots, μ_n be the roots of $g(x) = 0$ and suppose that

$$\max\{|\lambda_j - \mu_j| : j = 1, 2, \dots, n\} < \delta, \quad (*)$$

where

$$\delta = \frac{Mr}{nN_0(m+r)^{N_0-1}},$$

with

$$m = \max\{1, \lambda_1\}, \quad N_0 = \left\lceil \frac{\ln(2(n-1))}{\ln(1/\ell)} \right\rceil.$$

Then $|\mu_1|$ is greatest among all the $|\mu_j|$, and, since $g(x)$ has real coefficients, μ_1 is real and, since λ_1 is positive, so is μ_1 . Let

$$S_k = \mu_1^k + \dots + \mu_n^k.$$

Then $|s_k - S_k| \leq \sum_{i=1}^n |\lambda_i^k - \mu_i^k|$. Now

$$\begin{aligned} |\lambda_i^k - \mu_i^k| &= |\lambda_i - \mu_i| |\lambda_i^{k-1} + \lambda_i^{k-2}\mu_i + \dots + \mu_i^{k-1}| \\ &< \delta k (\lambda_1 + r)^{k-1}. \end{aligned}$$

Suppose that $k \geq N_0$. Then $S_k \geq (\lambda_1 - r)^k - (n-1)(\lambda_0 + r)^k = (\lambda_1 - r)^k (1 - (n-1)(\frac{\lambda_0 + r}{\lambda_1 - r})^k) > (\frac{1}{2})(\lambda_1 - r)^k > 0$. For $k \leq N_0$,

$$\begin{aligned} |s_k - S_k| &\leq \delta(1 + 2(\lambda_1 + r) + \dots + N_0(\lambda_1 + r)^{N_0-1}) \\ &< \delta n N_0 (m + r)^{N_0-1} = Mr \leq M. \end{aligned}$$

So $S_k \geq 0$, for all $k \geq 2$. Also, $S_1 = s_1 \geq 0$. This shows that if we can choose N so that the inequality

$$\max\{|\lambda_j - \mu_j| : j = 1, 2, \dots, n\} < \delta$$

holds for that δ , then the corresponding X_N will be a nonnegative matrix with spectrum $\lambda_1, \dots, \lambda_n$ and $N - n$ zeros. Now,

$$\begin{aligned} \max\{|\lambda_j - \mu_j| : j = 1, 2, \dots, n\} &\leq \\ &(\frac{16}{3\sqrt{3}})(\sum_{k=1}^n |p_k|(\frac{N^{k-1}}{(N-1)\dots(N-k+1)} - 1)\gamma^{n-k})^{1/n}. \end{aligned}$$

But

$$\frac{N^{k-1}}{(N-1)\dots(N-k+1)} - 1 \leq \frac{2n^2}{N}, \text{ if } N > n^2.$$

By definition, $\gamma = 2 \max\{1, |p_k|^{1/k}, k = 1, 2, \dots, n\}$. Hence

$$\max\{|\lambda_j - \mu_j| : j = 1, 2, \dots, n\} \leq (\frac{16\gamma}{3\sqrt{3}})(\frac{2n^3}{N})^{1/n}.$$

But $n^{1/n} \leq 3^{1/3}$. Hence

$$\max\{|\lambda_j - \mu_j| : j = 1, 2, \dots, n\} \leq \frac{16\gamma \cdot 2^{1/n}}{\sqrt{3}N^{1/n}} \leq \delta,$$

provided

$$N \geq \frac{2^{4n+1}\gamma^n}{3^{n/2}\delta^n} = \frac{2(16\gamma n N_0(m+r)^{N_0-1})^n}{3^{n/2}M^n r^n}.$$

This gives the required bound.

There are variations of the Ostrowski bound, some using the Bombieri norm in place of the ℓ_2 one, available though the work of Beauzamy [1], Galantai and Hegedus [7], and these may lead to better bounds for N in certain circumstances. However, the main interest is that such a bound exists, and the general form it has.

When the Perron root $\lambda_1 = 1$, a nonnegative matrix A with the given nonzero spectrum can be made stochastic. In this case r and ℓ are measures of the spectral gap, which control the rate at which the powers of A converge to the stationary state of the corresponding Markov process. The size of N_0 is inversely related to r and ℓ .

The number M measures how close to zero the power sums can get, and we see its appearance (as M^n) in the denominator of the bound.

We conclude with an example involving the realization of a spectrum with three nonzero entries.

Example 5 $\sigma = (\rho, \exp(\frac{\pi i}{10}), \exp(\frac{-\pi i}{10}))$ has all its power sums positive if $\rho > \sqrt[9]{2 \cos(\pi/10)} = 1.07\dots$. If we take $\rho = 1.1$, and carry out the algorithm, we find that σ with 125 zeros added is the spectrum of an 128×128 nonnegative matrix of the form of X above. The least number of zeros required to be added to σ to ensure realizability does not appear to be known in this case.

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